



On a leader election algorithm: Truncated geometric case study



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ABSTRACT

Recent work of Kalpathy and Mahmoud (in press) gives very general results for a broad class of fair leader election algorithms. They study the duration of contestants, i.e., the number of rounds a randomly selected contestant stays in the competition and another parameter for the associated tree structure. They present a unifying treatment for leader election algorithms, and they show how perpetuities naturally come about. Their theory, however, produces only trivial asymptotic results for the duration of election for some distributions, such as a truncated geometric distribution. In the case of a truncated geometric distribution, the limiting distribution of the duration of contestants is degenerate, and the method of Kalpathy and Mahmoud (in press) does not yield the precise asymptotics. The goal of this short note is to use an alternative method – namely, the q -series methodology – to make a very precise asymptotic analysis of the rate of decay of the mean and the variance of the duration of the election.

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1. Background

Randomized divide and conquer algorithms have several manifestations. In particular, leader election problems are an interesting class, because randomized elections have a rich mathematical and algorithmic history. Prodinge (1993) precisely analyzed the average behavior of several characteristic properties for a certain leader election algorithm that flips unbiased coins. He coined the terminology *incomplete trie* for the tree structure underlying the elimination process. Using analytic methods, the exact and asymptotic average for the size of the tree, i.e., the number of nodes, the depth (also known as the height in the literature) or the number of rounds, and the cost (measured in terms of the total number of coin flips) were obtained. Fill et al. (1996) used analytic and probabilistic methods to obtain the oscillating distribution of the height of a random incomplete trie constructed using unbiased coins. Janson and Szpankowski (1997) analyzed the height for biased coins using analytic techniques. Mohamed (2006) also investigated the biased-case scenario for the height, but used probabilistic methods. More recently, Louchard and Prodinge (2009) used analytic methods to study the number of rounds in a coin flipping selection algorithm that occurs in the presence of a demon (who randomly eliminates some contestants). Louchard et al. (2012), complementing the previous paper, precisely analyzed the distribution and all moments of the number of survivors in a selection process that occurs in the presence of a demon. Louchard et al. (2011) studied another variant called the Swedish leader election protocol; they analyzed several parameters, e.g., the probability of success, the expected

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number of rounds, the expected number of players still playing by the time the protocol fails, etc., using analytic methods. Kalpathy et al. (2011) used analytic and probabilistic techniques to study a leader election algorithm using biased coins for the duration a particular player survives in the competition and the total cost involved in the selection process.

Some more general, broad frameworks (which unify some of the above results) have also been proposed, for the study of leader election algorithms. One such framework was proposed by Janson et al. (2008), giving a theory for the cost associated with the number of rounds (equivalently, the height of the underlying incomplete tree). Another framework was proposed by Kalpathy et al. (2013), giving the number of survivors in a broad class of fair leader election algorithms, using the theory of probability metrics. That framework yields product random variables as the limit distribution. The classical example of leader election is via binomial splitting (where candidates advance in rounds of coin flipping). The binomial splitting protocol is only one of many possible strategies. More recently, Kalpathy and Mahmoud (in press) provided a broader framework to cover several splitting strategies of interest at once. They obtain very general results for general strategies, like uniform splitting, ladder tournaments, etc. They study (a) the duration of a particular contestant, i.e., the number of rounds a randomly selected contestant stays in the competition, and (b) the total cost of selection. Using probability metrics and the contraction method, they present a unifying treatment for leader election algorithms, and they show how perpetuities naturally come about (see Kalpathy (2013) for a detailed discussion).

Kalpathy and Mahmoud (in press) look at a set of mild sufficient conditions, which are easily met by most practical choices of splitting protocols, such as binomial, uniform, ladders, power laws, etc. Their theory, however, produces only trivial asymptotic results for the duration of election for some distributions, such as a truncated geometric distribution. In the case of a truncated geometric distribution, the limiting distribution of the duration of contestants is degenerate, and the method of Kalpathy and Mahmoud (in press) is insufficient for obtaining precise asymptotics.

Thus, the truncated geometric distribution corresponds to a type of leader election example in which sharper tools are needed, to give necessary precision to the asymptotics. We opted to use the q -series methodology, because it provides sufficient sharpness, and it can produce such exact asymptotic results.

We also emphasize that this type of asymptotic analysis is usually handled by breaking the classification of the parameters p and $q = 1 - p$ into three separate cases: (1) $p = q = 1/2$; (2) $\frac{\ln p}{\ln q}$ is a rational number; (3) $\frac{\ln p}{\ln q}$ is an irrational number. Our analysis is sufficiently precise to handle all p 's and q 's.

2. Asymptotic notation

We use the Θ notation from Knuth (1976), which describes the asymptotic growth of functions; we quote from page 20:

“ $\Theta(f(n))$ denotes the set of all $g(n)$ such that there exist positive constants C, C' , and n_0 with $Cf(n) \leq g(n) \leq C'f(n)$ for all $n \geq n_0$ ”.

This notation is quite standard in asymptotic analysis of algorithms. We also use another standard notation to indicate that f grows at a strictly smaller rate than g , namely, $f(n) = o(g(n))$ if, for all $C > 0$, there exists n_0 (depending on C) such that $|f(n)| \leq C|g(n)|$ for all $n \geq n_0$.

3. Duration for truncated geometric splitting

Let D_n be the duration of a specific contestant (say Bob) when starting the election with n contestants, i.e., the number of rounds Bob participated. Note that this is equivalent to the depth of the underlying incomplete tree. We set $D_0 = D_1 = 0$. At the start of a round, if n contestants are present, then K_n contestants advance to the next round. We handle the case where K_n is a truncated geometric random variable with parameters p and $q := 1 - p$. We have

$$\mathbf{P}(K_n = \ell) = cpq^\ell, \quad \text{for some constant } c.$$

Since $1 = \sum_{\ell=0}^n \mathbf{P}(K_n = \ell) = c \sum_{\ell=0}^n pq^\ell$, it follows that $c = \frac{1}{1-q^{n+1}}$. Thus the mass of K_n is

$$\mathbf{P}(K_n = \ell) = \frac{pq^\ell}{1 - q^{n+1}}, \quad \text{for } \ell = 0, 1, \dots, n.$$

Suppose we conduct a leader election among n contestants, in which a fair selection of a subset of contestants of a random size K_n advances to the next round, and the algorithm is applied recursively on that subset, till one leader or none is elected. We get the following result.

Theorem 3.1. *Let D_n be the duration of a specific contestant that begins with n participants. If the number of contestants who advance to the next round is a truncated geometric random variable with parameters p and q , then the first and second moments of D_n are*

$$\mathbf{E}[D_n] = 1 + \frac{1}{n} \sum_{k=2}^n \frac{kpq^k}{1 - q^k},$$

and

$$\begin{aligned} \mathbf{E}[D_n^2] &= 1 + \frac{1}{n} \left[3pn \sum_{k=1}^n \frac{q^k}{1-q^k} - 3p \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{q^k}{1-q^k} - 3q \right] - \frac{1}{n} \left[2pq \sum_{k=1}^n \frac{q^k}{1-q^k} \right] \\ &+ \frac{1}{n} \left[np^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2} + np^2 \left(\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right)^2 - 2p^2 \sum_{j=2}^n \sum_{\ell=1}^{j-1} \sum_{k=\ell}^n \frac{q^k}{(1-q^k)} \frac{q^\ell}{(1-q^\ell)} \right] \\ &- 2p^2 \sum_{k=n+1}^{\infty} \sum_{\ell=1}^k \frac{q^k}{1-q^k} \frac{q^\ell}{1-q^\ell}. \end{aligned}$$

Corollary 3.1. *If the number of contestants who advance to the next round of an election (with n initial participants) is a truncated geometric random variable with parameters p and q , then the duration D_n of a specific contestant has variance*

$$\begin{aligned} \text{Var}[D_n] &= \frac{1}{n} \left[3pn \sum_{k=1}^n \frac{q^k}{1-q^k} - 3p \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{q^k}{1-q^k} - 3q \right] - \frac{1}{n} \left[2pq \sum_{k=1}^n \frac{q^k}{1-q^k} \right] \\ &+ \frac{1}{n} \left[np^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2} + np^2 \left(\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right)^2 - 2p^2 \sum_{j=2}^n \sum_{\ell=1}^{j-1} \sum_{k=\ell}^n \frac{q^k}{(1-q^k)} \frac{q^\ell}{(1-q^\ell)} \right] \\ &- 2p^2 \sum_{k=n+1}^{\infty} \sum_{\ell=1}^k \frac{q^k}{1-q^k} \frac{q^\ell}{1-q^\ell} - \frac{2}{n} \sum_{k=2}^n \frac{kpq^k}{1-q^k} - \frac{1}{n^2} \left(\sum_{k=2}^n \frac{kpq^k}{1-q^k} \right)^2. \end{aligned}$$

Theorem 3.1 contains exact characterizations of the first and second moments of the duration of a specific contestant. We also are interested in the asymptotic properties of the moments. We note that $a_p := \sum_{k=2}^{\infty} \frac{kpq^k}{1-q^k}$ is a constant (depending only on p). Thus, $\mathbf{E}[D_n] = 1 + \mathcal{O}(1/n)$, and moreover, $\lim_{n \rightarrow \infty} (\mathbf{E}[D_n] - 1)(n) = a_p$. The plot of a_p is given in [Fig. 1](#).

For the second moment expression, we first characterize the contribution of one (relatively) small term, namely,

$$2p^2 \sum_{k=n+1}^{\infty} \sum_{\ell=1}^k \frac{q^k}{1-q^k} \frac{q^\ell}{1-q^\ell} = o(1/n).$$

Also if we define

$$\begin{aligned} b_p &:= \lim_{n \rightarrow \infty} \left[3pn \sum_{k=1}^n \frac{q^k}{1-q^k} - 3p \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{q^k}{1-q^k} - 3q \right], \\ c_p &:= \lim_{n \rightarrow \infty} \left[2pq \sum_{k=1}^n \frac{q^k}{1-q^k} \right], \\ d_p &:= \lim_{n \rightarrow \infty} \left[np^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2} + np^2 \left(\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right)^2 - 2p^2 \sum_{j=2}^n \sum_{\ell=1}^{j-1} \sum_{k=\ell}^n \frac{q^k}{(1-q^k)} \frac{q^\ell}{(1-q^\ell)} \right], \end{aligned}$$

then b_p, c_p, d_p are constants (depending only on p). Thus, $\mathbf{E}[D_n^2] = 1 + \mathcal{O}(1/n)$, and moreover, we note that $\lim_{n \rightarrow \infty} (\mathbf{E}[D_n^2] - 1)(n) = b_p - c_p + d_p$. The plots of b_p, c_p , and d_p are given in [Figs. 2–4](#), respectively.

4. Proof of Theorem 3.1

Given the value of K_n , we can set up a conditional equation for the value of D_n . We have:

- If $n = 0$, then $K_0 = 0$ (always), because $D_0 = 0$ in this case. (The rationale is: nobody advances to the next round, so no additional levels were needed in the election.)
- If $n = 1$, then K_1 is 0 or 1, because $D_1 = 0$ in this case. (The rationale is: only one person is present at the start of the round, so no additional levels were needed in the election.)
- If $n \geq 2$, then K_n is between 0 and n (inclusive). Then:
 - Given the value of K_n , Bob is one of the K_n participants with probability K_n/n , so the *conditional distribution* of D_n (under these conditions) is the same as the unconditional distribution of $1 + D_{K_n}$. One round was used, and K_n contestants will participate in the next round, including the specific contestant (Bob).
 - Given the value of K_n , Bob fails to be one of the K_n participants with probability $1 - K_n/n$, so the *conditional distribution* of D_n (under these conditions) is 1. One round was used, and then the specific contestant (Bob) is removed from the rest of the election.

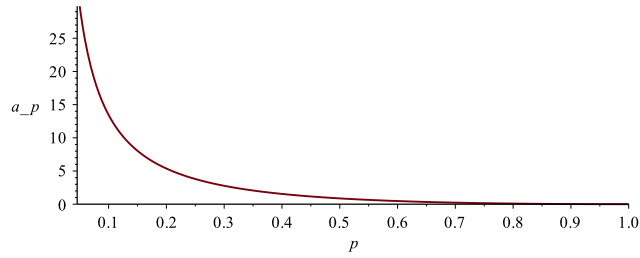


Fig. 1. Plot of $a_p := \sum_{k=2}^{\infty} \frac{kpq^k}{1-q^k}$ for $1/20 \leq p \leq 1$.

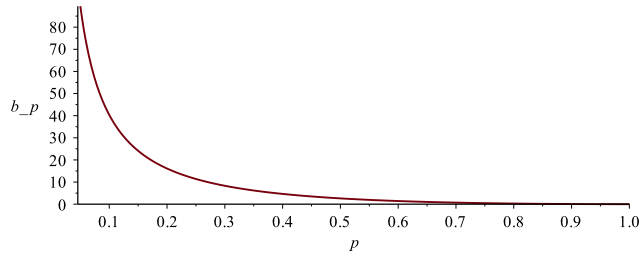


Fig. 2. Plot of $b_p := \lim_{n \rightarrow \infty} \left[3pn \sum_{k=1}^n \frac{q^k}{1-q^k} - 3p \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{q^k}{1-q^k} - 3q \right]$ for $1/20 \leq p \leq 1$.

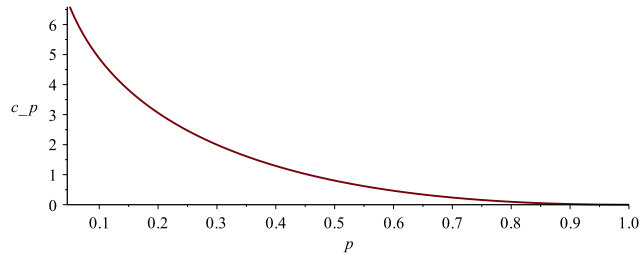


Fig. 3. Plot of $c_p := \lim_{n \rightarrow \infty} \left[2pq \sum_{k=1}^n \frac{q^k}{1-q^k} \right]$ for $1/20 \leq p \leq 1$.

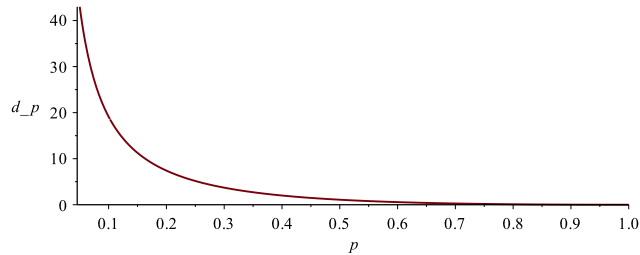


Fig. 4. Plot of $d_p := \lim_{n \rightarrow \infty} \left[np^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2} + np^2 \left(\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right)^2 - 2p^2 \sum_{j=2}^n \sum_{\ell=1}^{j-1} \sum_{k=\ell}^n \frac{q^k}{(1-q^k)} \frac{q^\ell}{(1-q^\ell)} \right]$ for $1/20 \leq p \leq 1$.

The rest of the proof is developed using the q -series methodology. We define $\phi_n(t) := \mathbf{E}[e^{tD_n}]$. We have $\phi_0(t) = \phi_1(t) = 1$, and then for $n \geq 2$, we have the recursion

$$\begin{aligned} \phi_n(t) &= \mathbf{E}[e^{tD_n}] \\ &= \sum_{\ell=0}^n \mathbf{E}[e^{tD_n} \mid K_n = \ell] \mathbf{P}(K_n = \ell) \\ &= \sum_{\ell=0}^n \left(\mathbf{E}[e^{t(1+D_\ell)}] \frac{\ell}{n} + \mathbf{E}[e^{t(1)}] \left(1 - \frac{\ell}{n} \right) \right) \mathbf{P}(K_n = \ell) \\ &= e^t \sum_{\ell=0}^n \left(\mathbf{E}[e^{tD_\ell}] \frac{\ell}{n} + 1 - \frac{\ell}{n} \right) \mathbf{P}(K_n = \ell). \end{aligned}$$

Thus, for $n \geq 2$, we have

$$\begin{aligned} n\phi_n(t) &= e^t \sum_{\ell=0}^n (\mathbf{E}[e^{tD_\ell}] \ell + n - \ell) \mathbf{P}(K_n = \ell) \\ &= ne^t \phi_n(t) \frac{pq^n}{1 - q^{n+1}} + e^t \sum_{\ell=0}^{n-1} (\phi_\ell(t) \ell + n - \ell) \frac{pq^\ell}{1 - q^{n+1}}. \end{aligned}$$

So, for $n \geq 2$, we have

$$(1 - q^{n+1})n\phi_n(t) = ne^t \phi_n(t) pq^n + e^t \sum_{\ell=0}^{n-1} (\phi_\ell(t) \ell + n - \ell) pq^\ell,$$

and thus,

$$n(1 - q^n(q + e^t p))\phi_n(t) = e^t \sum_{\ell=0}^{n-1} (\phi_\ell(t) \ell + n - \ell) pq^\ell.$$

Hence, for $n \geq 3$, we have

$$(n - 1)(1 - q^{n-1}(q + e^t p))\phi_{n-1}(t) = e^t \sum_{\ell=0}^{n-2} (\phi_\ell(t) \ell + n - 1 - \ell) pq^\ell.$$

It follows that, for $n \geq 3$,

$$n(1 - q^n(q + e^t p))\phi_n(t) - (n - 1)(1 - q^{n-1}(q + e^t p))\phi_{n-1}(t) = e^t (\phi_{n-1}(t)(n - 1)) pq^{n-1} + e^t (1 - q^n).$$

Rearranging the above equation, we get

$$n(1 - q^n(q + e^t p))\phi_n(t) = (n - 1)\phi_{n-1}(t)(1 - q^n) + e^t (1 - q^n).$$

Consequently for $n \geq 3$, we have

$$\phi_n(t) = \frac{(n - 1)(1 - q^n)}{n(1 - q^n(q + e^t p))} \phi_{n-1}(t) + \frac{e^t (1 - q^n)}{n(1 - q^n(q + e^t p))}.$$

For $n \geq 3$, we obtain the solution

$$\phi_n(t) = \frac{2}{n} \prod_{i=3}^n \frac{1 - q^i}{1 - q^i(q + e^t p)} \phi_2(t) + \sum_{j=3}^n \frac{e^t}{n} \prod_{i=j}^n \frac{1 - q^i}{1 - q^i(q + e^t p)},$$

which is obtained by iteration. Also, we know that

$$\phi_2(t) = e^t \frac{p(1 + q)}{1 - q^2(q + pe^t)} = e^t \frac{1 - q^2}{1 - q^2(q + e^t p)}.$$

Therefore, for $n \geq 3$, we find

$$\phi_n(t) = \frac{2e^t}{n} \prod_{i=2}^n \frac{1 - q^i}{1 - q^i(q + e^t p)} + \sum_{j=3}^n \frac{e^t}{n} \prod_{i=j}^n \frac{1 - q^i}{1 - q^i(q + e^t p)}.$$

Using half of the first term, to serve as the $j = 2$ term in the second summation, it follows that, for $n \geq 3$,

$$\phi_n(t) = \frac{e^t}{n} \prod_{i=2}^n \frac{1 - q^i}{1 - q^i(q + e^t p)} + \sum_{j=2}^n \frac{e^t}{n} \prod_{i=j}^n \frac{1 - q^i}{1 - q^i(q + e^t p)}. \quad (1)$$

Note that the equation above holds for $n = 2$ too.

We define the q -Pochhammer symbol as

$$(x)_n := \prod_{j=0}^{n-1} (1 - xq^j).$$

To simplify the moment generating function, we note

$$\prod_{i=j}^n (1 - q^i) = \frac{(q)_n}{\prod_{k=1}^{j-1} (1 - q^k)}, \quad \text{and} \quad \prod_{i=j}^n (1 - q^i(q + e^t p)) = \frac{(q + e^t p)_{n+1}}{\prod_{k=0}^{j-1} (1 - q^k(q + e^t p))}. \quad (2)$$

Applying (2) to the expression for $\phi_n(t)$ in (1), we get, for $n \geq 2$,

$$\phi_n(t) = \frac{e^t}{n} \frac{(q)_n}{(q + e^t p)_{n+1}} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right).$$

Since $(q)_n = (q)_\infty / (q^{n+1})_\infty$ and $(q + e^t p)_{n+1} = (q + e^t p)_\infty / ((q + e^t p) q^{n+1})_\infty$, this yields, for $n \geq 2$,

$$\phi_n(t) = \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right). \tag{3}$$

Using $\mathbf{E}[D_n] = \lim_{t \rightarrow 0} \frac{d}{dt} \phi_n(t)$, we obtain

$$\begin{aligned} \mathbf{E}[D_n] &= \lim_{t \rightarrow 0} \left(\frac{d}{dt} \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right) \right) \\ &= \lim_{t \rightarrow 0} \left[\left(\frac{d}{dt} e^t \right) \frac{1}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right) \right. \\ &\quad + \left(\frac{d}{dt} ((q + e^t p) q^{n+1})_\infty \right) \frac{e^t}{n} \frac{(q)_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right) \\ &\quad + \left(\frac{d}{dt} \frac{1}{(q + e^t p)_\infty} \right) \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty} \left(\frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right) \\ &\quad \left. + \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} \left(\frac{d}{dt} \frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \frac{d}{dt} \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)} \right) \right]. \end{aligned}$$

The four terms in the expression on the last line simplify to these analogous four terms:

$$\mathbf{E}[D_n] = 1 + \sum_{k=0}^{\infty} \frac{-p q^{n+1+k}}{1 - q^{n+1+k}} + \sum_{k=1}^{\infty} \frac{p q^k}{1 - q^k} + \frac{1}{n} \left(\frac{-q p}{1 - q} + \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{-p q^k}{1 - q^k} \right).$$

Equivalently,

$$\mathbf{E}[D_n] = 1 + \frac{1}{n} \sum_{k=2}^n \frac{k p q^k}{1 - q^k}.$$

We now derive the second moment. For succinctness of notation, we write $B_n(t)$ for the following expression:

$$B_n(t) := \frac{\prod_{k=0}^1 (1 - q^k (q + e^t p))}{1 - q} + \sum_{j=2}^n \frac{\prod_{k=0}^{j-1} (1 - q^k (q + e^t p))}{\prod_{k=1}^{j-1} (1 - q^k)}.$$

Starting with the expression for $\phi_n(t)$ in (3), and using $\mathbf{E}[D_n^2] = \lim_{t \rightarrow 0} \frac{d^2}{dt^2} \phi_n(t)$, we obtain

$$\begin{aligned} \mathbf{E}[D_n^2] = \lim_{t \rightarrow 0} & \left[\left(\frac{d^2}{dt^2} e^t \right) \frac{1}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n(t) + \left(\frac{d^2}{dt^2} ((q + e^t p) q^{n+1})_\infty \right) \frac{e^t}{n} \frac{(q)_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n(t) \right. \\ & + \left(\frac{d^2}{dt^2} \frac{1}{(q + e^t p)_\infty} \right) \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty} B_n(t) + \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n''(t) \\ & + 2 \left(\frac{d}{dt} e^t \right) \left(\frac{d}{dt} ((q + e^t p) q^{n+1})_\infty \right) \frac{1}{n} \frac{(q)_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n(t) \\ & + 2 \left(\frac{d}{dt} ((q + e^t p) q^{n+1})_\infty \right) \left(\frac{d}{dt} \frac{1}{(q + e^t p)_\infty} \right) \frac{e^t}{n} \frac{(q)_\infty}{(q^{n+1})_\infty} B_n(t) \\ & + 2 \left(\frac{d}{dt} e^t \right) \left(\frac{d}{dt} \frac{1}{(q + e^t p)_\infty} \right) \frac{1}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty} B_n(t) \\ & + 2 \left(\frac{d}{dt} e^t \right) \frac{1}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n'(t) \\ & + 2 \left(\frac{d}{dt} ((q + e^t p) q^{n+1})_\infty \right) \frac{e^t}{n} \frac{(q)_\infty}{(q^{n+1})_\infty (q + e^t p)_\infty} B_n'(t) \\ & \left. + 2 \left(\frac{d}{dt} \frac{1}{(q + e^t p)_\infty} \right) \frac{e^t}{n} \frac{(q)_\infty ((q + e^t p) q^{n+1})_\infty}{(q^{n+1})_\infty} B_n'(t) \right]. \end{aligned}$$

We emphasize that the necessary computations to simplify this expression are non-trivial. The authors diligently tried to use computational symbolic algebra, but MAPLE was unable to simplify most portions of the above expression.

Instead, we worked by hand to separate the expression into three portions: the parts without poles, the parts that potentially could yield single-order poles (i.e., multiples of $\frac{1}{(1-(q+e^t p))}$), and the parts that potentially could yield double-order poles (i.e., multiples of $\frac{1}{(1-(q+e^t p))^2}$). As must be the case, all of the parts that could potentially correspond to simple or double poles simplified to 0, as we knew that they must.

After a tremendous amount of simplification, we see that

$$\begin{aligned} \mathbf{E}[D_n^2] = 1 + \frac{1}{n} & \left[3pn \sum_{k=1}^n \frac{q^k}{1-q^k} - 3p \sum_{j=2}^n \sum_{k=1}^{j-1} \frac{q^k}{1-q^k} - 3q \right] - \frac{1}{n} \left[2pq \sum_{k=1}^n \frac{q^k}{1-q^k} \right] \\ & + \frac{1}{n} \left[np^2 \sum_{k=1}^{\infty} \frac{q^{2k}}{(1-q^k)^2} + np^2 \left(\sum_{k=1}^{\infty} \frac{q^k}{1-q^k} \right)^2 - 2p^2 \sum_{j=2}^n \sum_{\ell=1}^{j-1} \sum_{k=\ell}^n \frac{q^k}{(1-q^k)} \frac{q^\ell}{(1-q^\ell)} \right] \\ & - 2p^2 \sum_{k=n+1}^{\infty} \sum_{\ell=1}^k \frac{q^k}{1-q^k} \frac{q^\ell}{1-q^\ell}. \end{aligned}$$

The above first and second moment computations illustrate the kinds of results that one obtains with a truncated geometric splitting protocol using q -series analysis.

5. Concluding remarks

This short note provides an interesting example to the leader election problem studied in Kalpathy and Mahmoud (in press), where their theory is insufficient to give an informative limit. Their theory produces only trivial asymptotic results for the duration of the election, when the splitting protocol K_n follows a truncated geometric distribution. In this scenario, K_n/n converges in distribution to 0. Intuitively speaking, for the truncated distribution, much of the mass is at the beginning and it has small tail probabilities. It is as if an overwhelming majority of the contestants get eliminated at the start. For such distributions, the exact calculations are more interesting than the asymptotics, and the q -series methodology gives a precise way to produce exact results.

6. Open questions

Another example where the leader election problem studied in Kalpathy and Mahmoud (in press) produces trivial asymptotic results for the duration is when the splitting protocol K_n follows a truncated Poisson distribution. In this scenario, K_n/n converge in distribution to 0, just like the truncated geometric case. Hence, the exact calculations are more interesting than the asymptotics. Proceeding on similar lines, it would be interesting to investigate other truncated discrete distributions such as a truncated Poisson splitting protocol.

The framework proposed by Kalpathy et al. (2013) gives the number of survivors in a broad class of fair leader election algorithms, after t number of election rounds, using the theory of probability metrics. We found it challenging to find the exact moment generating function for the number of survivors with truncated geometric splitting protocol. An intuitive reason for this difficulty is because of the additional layers of complexity as t grows.

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